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State-Space Representation of Aerodynamic Characteristics of an Aircraft at High Angles of Attack

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Mathematical modeling of unsteady aerodynamic forces and moments plays an important role in aircraft dynamics investigation and stability analysis at high angles of attack. In this article the state-space representation of aerodynamic forces and moments for unsteady aircraft motion is proposed. Consideration of separated flow about an airfoil and flow with vortex breakdown about a slender delta wing gives the base for mathematical modeling using internal variables describing the flow state. Coordinates of separation points or vortex breakdown can be taken, e.g., as internal state-space variables. These variables are governed by some differential equations. Within the framework of the proposed mathematical model it is possible to achieve good agreement with different experimental data obtained in water and wind tunnels. These high angle-of-attack experimental results demonstrate considerable dependence of aerodynamic loads on motion time history.

Nomenclature

A = wing aspect ratio, $(4b^2)/(S)$
b = semispan
C_L = lift coefficient
$C_{l,m,n}$ = rolling-, pitching-, and yawing-moment coefficient
$C_{X,Y,Z}$ = longitudinal-, lateral-, and vertical-force coefficient
c = root chord of a delta wing
$\bar{c}$ = wing mean aerodynamic chord
p, q, r = roll, pitch, and yaw rates
S = reference wing area
s = Laplace variable
t = time
V = airspeed
x = state-space internal dynamic variable
α = angle of attack
β = sideslip angle
$\delta_a, \delta_e, \delta_r$ = aileron, elevator, and rudder deflections
τ_1, τ_2 = characteristic times
ϕ = roll angle
ω = frequency of oscillations

Subscripts

l = left
r = right
0 = steady-state conditions

Superscripts

att = component due to the attached flow
fo = forced oscillations method results
nl = nonlinear function
st = static dependency
T = transposed vector
vor = component due to the vortex breakdown influence

Aerodynamic Derivatives

$$C_{a_q} = \frac{\partial C_a}{\partial(q\bar{c}/V)} \quad C_{a_\alpha} = \frac{\partial C_a}{\partial \alpha} \quad C_{a_\delta} = \frac{\partial C_a}{\partial \delta}$$

$$C_{a_p} = \frac{\partial C_a}{\partial(pb/V)} \quad C_{a_r} = \frac{\partial C_a}{\partial(rb/V)} \quad C_{a_\beta} = \frac{\partial C_a}{\partial \beta}$$

$$C_{m_{\dot\alpha}} = \frac{\partial C_m}{\partial(\dot\alpha\bar{c}/V)}$$

where $a = L, X, Y, Z, l, m$, or n.

Introduction

As the high angles-of-attack region becomes more accessible for modern aircraft, the problem of adequate mathematical modeling of aerodynamic characteristics at separated and vortex breakdown flow conditions arises. Mathematical modeling of aerodynamic characteristics is needed for flight dynamics simulation and stability analysis of aircraft motion at high angles of attack, and is very important for solving problems of flight safety and the study of critical aircraft regimes such as stall and spin.

In flight dynamics there are different methods of aerodynamic coefficients modeling. In many practical cases the aerodynamic forces and moments are approximated by linear terms in their Taylor series expansion, a well known approach leading to stability and control derivatives.[1,2] This method is sufficiently accurate for attached flows at low angles of attack. For example, the representation of pitch moment coefficient has the following form:

$$C_m = C_{m_0}(\alpha) + C_{m_q}(q\bar{c}/V) + C_{m_{\dot\alpha}}(\dot\alpha\bar{c}/V) + C_{m_{\delta_e}}\delta_e$$

But in the region of α where separated and vortex flow is developed this representation cannot be used. The values of unsteady derivatives are strongly dependent on the amplitude and frequency of aircraft oscillations.[3]

Another more correct approach used to obtain the airloads on an aircraft undergoing an arbitrary motion is the indicial response method in conjunction with the superposition principle[1,4]

$$C = \int_0^t A(t - \tau)\dot{h}\, d\tau$$

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where $C = (C_X, C_Y, C_Z, C_l, C_m, C_n)^T$ is the combined vector of total aerodynamic force and moment coefficients, $A = \{A_{C_j}^{h_i}\}$ is a matrix of indicial response functions for stepwise variation of kinematic parameters, such as angle of attack and sideslip, roll, pitch and yaw rates, control surfaces deflections, which form a vector $h = (\alpha, \beta, p, q, r, \delta_a, \delta_e, \delta_r)^T$.

This approach is certainly efficient, but it is difficult to combine this functional representation with the equations of an aircraft motion, which are written in the differential equation form. Such an approach can also be extended to a nonlinear case corresponding to the separated flow conditions.[4] But in this case the use of such a description becomes significantly complicated.

In some cases Fourier functional analysis can also be used for unsteady aerodynamic modeling.[5]

The description of the variation of nonlinear aerodynamic characteristics associated with the motion time history is possible also by utilizing the ordinary differential equations.[6,7] In order to describe aerodynamic characteristics measured experimentally $C(t) \in R^6$ the following type input-output dynamical system is used:

$$D_i\left(C, \frac{dC}{dt}, \ldots, \frac{d^kC}{dt^k}\right) = N_i\left(h, \frac{dh}{dt}, \ldots, \frac{d^mh}{dt^m}\right)$$

$$C \in R^6, \quad h \in R^8, \quad i = 1, \ldots, 6$$

where D_i, N_i are some smooth nonlinear functions, to be defined by the fitting technique.

This method is convenient for solving problems of flight dynamics since the inclusion of unsteady aerodynamics in the above form leads only to an increase in problem dimension, which retains the possibility of investigating motion stability by means of classical methods.

In this article this approach is explored by means of introducing internal variables for describing the state of the flow (the so-called state-space representation[8,9]). The state is a very important concept for the description of any dynamical system, e.g., of the aircraft aerodynamics. It gives the information required for the determination of the instant and future values of the aerodynamic characteristics. The internal state variables can be taken either in a formal way or can have the well-defined physical meaning.

There is a close relation between the state-space representation and the use of the indicial response function method. In the linear case the indicial response function can be approximated by a finite number of the exponential functions. This approximation is equivalent to the utilization of the set of first-order differential equations. The resulting first-order differential equations, describing the unsteady aerodynamics, can easily be added to the aircraft motion equations.

The state-space representation of unsteady airfoil behavior in the linear case of attached flow was presented in Ref. 9, where this approach was used to describe only vortex wake time-lag effects.

At high angles of attack the separated flow around the airfoil has some additional dynamical properties such as the boundary-layer convection lag, the boundary-layer improvement effects, moving separation point effect, etc.[10] Some of these flow phenomena influence the conditions of the appearance or disappearance of the flow separation. Other ones are connected with the separated flow development and adjustment processes. The instant position of the separation point depends on all these effects. The values of the total aerodynamic force and moment depend in their turn on the kinematic parameters of the motion and the position of flow separation. Thus, the separation point position can be taken as an internal dynamical variable in this case. To obtain the mathematical model the appropriate dynamic equations for governing the separation point behavior depending on the kinematic parameters of airfoil motion are needed, as well as the total aerodynamic load dependency on the kinematic parameters and the separation point position.

The similar aerodynamic processes guide the separated vortex flow around a low aspect ratio delta wing. In this case the vortex breakdown point positions can be taken as internal dynamical variables. Although the aerodynamic processes for an airfoil and a delta wing are quite different, the dynamic features of the aerodynamic loads are very similar.[11]

Taking into account the above features, one can propose the mathematical model where the internal dynamical variables (vector x) approximately describes the state of separated and vortex flow about an aircraft. These variables are simply the additional information required at a given instant of time to calculate the outputs (aerodynamic forces and moments—vector C) from the system inputs (vector h). This approach can be written by using the following type input-state-output dynamical system[12]:

$$\frac{dx}{dt} = f(x, h)$$
$$C = g(x, h) \tag{1}$$

To illustrate the possibility of using the proposed mathematical model, simple examples will be considered. The first example is the unsteady flow about an airfoil with a trailing-edge separation. The second is the unsteady flow about a slender delta wing with a vortex breakdown. On the basis of these examples the structure of the approximate mathematical model of unsteady aircraft aerodynamics is proposed.

Unsteady Flow About an Airfoil with Trailing-Edge Separation

The various fluid mechanical processes that produce the dynamic stall overshoot or undershoot of static airfoil characteristics are well known. The analytic formulas were obtained to outline these phenomena for the definite unsteady airfoil motions.[10] To simulate an arbitrary airfoil motion it is more desirable to obtain the mathematical model in the form of the dynamical system.

To obtain the simplest mathematical model one can consider the flow about an airfoil with trailing-edge separation without spilled vortex effects. This is valid for relatively slow variations of the airfoil incidence. Assume that the airfoil has a sufficient thickness for the development of flow separation in the vicinity of the trailing edge. A separated flow can be described by the nondimensional coordinate $x \in [0, 1]$, which gives the position of the separation point on the upper surface of the airfoil.[13]

The value $x = 1$ corresponds to attached flow, while $x = 0$ corresponds to leading-edge separation. One can consider that x is a specified internal state variable on which aerodynamic loads acting on the airfoil depend essentially.

At first it is important to obtain the relation between the airfoil aerodynamic coefficients and the input parameter—the angle of attack α and the internal variable—the separation point position x. The simplified approach based upon the assumption that the separated flow about the airfoil is modeled by Kirchgof's zone of constant pressure and the common assumption of linear cavitation theory[14] gives the following expressions for the airfoil lift and moment coefficients:

$$C_L^{\text{nl}}(\alpha, x) = (\pi/2)\sin\alpha(1 + \sqrt{x})^2$$
$$C_m^{\text{nl}}(\alpha, x) = (\pi/2)\sin\alpha(1 + \sqrt{x})^2 \left[\frac{5(1 - \sqrt{x})^2 + 4\sqrt{x}}{16}\right] \tag{2}$$

These expressions were obtained for quasisteady flow conditions. The moment coefficient was calculated with respect to airfoil nose.

If we know the steady-state dependence of the position of separation point on the variation of angle of attack $x_0(\alpha)$, then expressions (2) give us, e.g., the steady-state dependency of lift coefficient $C_L^{st}(\alpha) = C_L^{nl}[\alpha, x_0(\alpha)]$.

For unsteady flow conditions the aerodynamic loads of the airfoil will depend on the current value of the angle of attack $\alpha(t)$ and instantaneous separation point position $x(t)$ that can differ considerably from its stationary value $x_0(\alpha)$.

In the mathematical model the various unsteady fluid mechanics processes can be divided in two groups. The first group concerns the different quasisteady aerodynamic effects such as the circulation and boundary-layer convection lags, as well as the boundary-layer improvement effects. These effects influence the conditions of the flow separation and its reattachment. The resulting delay is approximately proportional to the angle of attack variation rate $\dot{\alpha}$. The quasisteady position of the separation point, e.g., can be expressed through the function $x_0(\alpha)$ by means of argument shift $x_0(\alpha - \tau_2\dot{\alpha})$, where the parameter τ_2 defines the total time delay of the above mentioned effects.

The second group of flow phenomena defines the transient aerodynamics effects, i.e., dynamic properties of the separated flow adjustment. Any disturbance of the steady separated flow without angle-of-attack variation will result in the appropriate adjustment or relaxation process leading to the steady state. One can describe this relaxation process in the simplest manner using the first-order differential equation.

Therefore, based on these assumptions one can use the following equation to describe the movement of the separation point for unsteady flow conditions:

$$\tau_1 \frac{dx}{dt} + x = x_0(\alpha - \tau_2\dot{\alpha}) \tag{3}$$

where τ_1 is the relaxation time constant.

The expressions Eqs. (2) and (3) form the closed mathematical model for the lift and moment coefficients of type Eq. (1). In this approach the effects of spilled vortex and wake vortex sheet are neglected.

The main qualitative features of the proposed mathematical model are presented in Fig. 1. The upper part of this figure shows the dependencies for the lift coefficient $C_L^{st}(\alpha)$ (solid line), $C_L^{nl}(\alpha, x)$ (dotted lines). The lower part shows the variation of the separation point position both for steady and unsteady conditions. The solid line defines the steady dependency $x_0(\alpha)$. The dashed lines show the variations of the separation point position for ramp motions with $\dot{\alpha} > 0$ and $\dot{\alpha} < 0$. The appropriate variations of the lift coefficient with

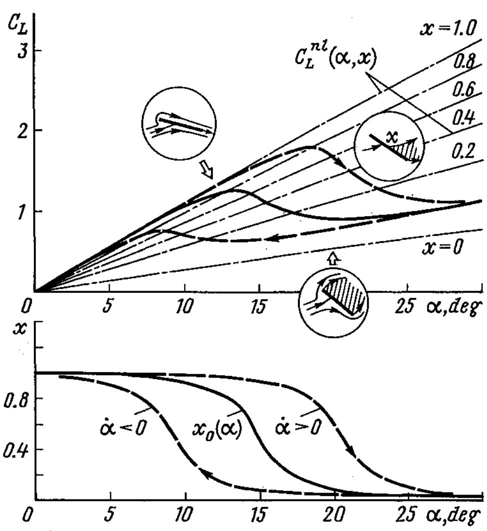

Fig. 1 Steady and unsteady lift of an airfoil with trailing-edge separation.

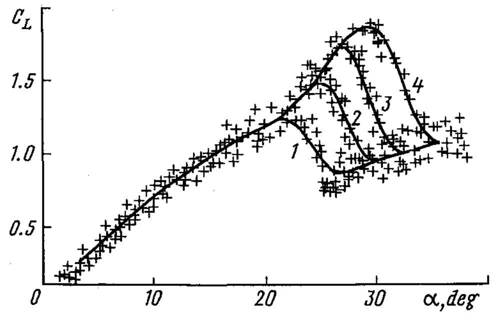

Fig. 2 Experimental data[15] and mathematical modeling results for a NACA 0015 airfoil rampwise motions ($\dot{\alpha}_1 < \dot{\alpha}_2 < \dot{\alpha}_3 < \dot{\alpha}_4$).

overshoot ($\dot{\alpha} > 0$) and undershoot ($\dot{\alpha} < 0$) are drawn also by the dashed lines in the upper part of Fig. 1.

Thus, the mathematical model, Eqs. (2) and (3), describes the main features observed in wind-tunnel experiments. This mathematical model contains two unknown parameters, τ_1 and τ_2, and an unknown function $x_0(\alpha)$, which have to be defined from the steady and unsteady experimental data. One possible way is to use the appropriate identification methods that take into account the structure of the mathematical model.

The proposed mathematical model also describes the airfoil oscillatory motion. For the case of small amplitude forced oscillations, Eqs. (2) and (3) can be used in the linearized form. The linearized equations obtain the relations between the in-phase and out-of-phase aerodynamic derivatives and the unknown parameters of the mathematical model

$$\begin{aligned} C_{L_\alpha}^{fo}(\alpha, \omega) &= \frac{\pi}{2}\cos\alpha[1 + \sqrt{x_0(\alpha)}]^2 \\ &\quad + \frac{\pi}{2}\sin\alpha \frac{1 + \sqrt{x_0(\alpha)}}{\sqrt{x_0(\alpha)}} \frac{dx_0}{d\alpha} \frac{(1 - \omega^2\tau_1\tau_2)}{(1 + \omega^2\tau_1^2)} \\ C_{L_{\dot{\alpha}}}^{fo}(\alpha, \omega) &= -\frac{\pi}{2}\sin\alpha \frac{1 + \sqrt{x_0(\alpha)}}{\sqrt{x_0(\alpha)}} \frac{dx_0}{d\alpha} \frac{\tau_1 + \tau_2}{(1 + \omega^2\tau_1^2)} \end{aligned} \tag{4}$$

The above expressions reveal, e.g., that the unsteady aerodynamic derivatives depend strongly on the frequency of oscillations in the angle-of-attack range where the flow separation is developed $[(dx_0/d\alpha) \neq 0]$. Equation (4) can also be used in the identification procedure. The more detailed discussion of this procedure will be presented in the next section devoted to the low aspect ratio delta wing, since the appropriate experimental data for the delta wing are available.

The example of the proposed mathematical model performance is demonstrated in Fig. 2. The wind-tunnel data from Ref. 15 for a NACA 0015 airfoil ramp motions with different pitch rates are shown by crosses ($\dot{\alpha}_1 < \dot{\alpha}_2 < \dot{\alpha}_3 < \dot{\alpha}_4$). The results of fitting them using the mathematical model are shown by solid lines. In this case the following values for time constants of Eq. (3) were obtained by the least squares technique $\tau_1 \approx 0.52(\bar{c}/V)$, $\tau_2 \approx 4.5(\bar{c}/V)$.

Unsteady Aerodynamic Characteristics of a Delta Wing with Vortex Breakdown

The similar mathematical model can be used also for unsteady aerodynamic characteristics of a slender delta wing with vortex breakdown.

Pitch Oscillations

First consider the case of symmetrical motion of the wing in the longitudinal plane. The nondimensional coordinate of vortex breakdown position can be taken as an internal state variable $x = x_L = x_R \in [0, 1]$ governed by the same differential Eq. (3).

To confirm this assumption, experimental results of steady and unsteady vortex breakdown visualization of a slender delta wing with aspect ratio $A = 1.5$ obtained in water tunnel[16] were simulated using Eq. (3).

The experimental dependency of vortex breakdown point position $x_0(\alpha)$ in steady flow conditions is shown in Fig. 3 by the bold solid line. This dependency is used in Eq. (3) as a function in the right side. In the case of unsteady wing motions the vortex breakdown point position can differ significantly from its steady-state position. The rampwise motions with the increase and decrease of the angle of attack ($\dot{\alpha} = 0.3V/c$—white circles, $\dot{\alpha} = 0.1V/c$—black triangles, $\dot{\alpha} = -0.05V/c$—black circles, $\dot{\alpha} = -0.1V/c$—white triangles) show that the processes of vortex breakdown and its recovery occur with the significant delay. The oscillatory motion with the relatively small amplitude 4.5 deg and frequency $\omega = 2.01V/c$ also demonstrate the significant delay. The experimental points in this case are marked by crosses. The appropriate results of the mathematical simulation made with the use of Eq. (3) ($\tau_1 = 1.5c/V$, $\tau_2 = 0.5c/V$) are shown by the solid lines. It is seen that the mathematical description is quite comprehensive.

For closing the mathematical model we must know the relationship between force and moment coefficients and the variation of α and x. For this purpose, e.g., the vortex suction analogy of Polhamus can be used.[17,18] In the case of the vortex breakdown on the upper surface of the wing the approximate expression for aerodynamic lift coefficient can be written in the following form:

$$C_L^{nl}(\alpha, x) = k_p \sin\alpha \cos^2\alpha + x^2 k_v \sin^2\alpha \cos\alpha \qquad (5)$$

where $k_p \approx \pi A/2$, $k_v \approx \pi$. To obtain this expression it was assumed that the vortex lift developed only on that part of the wing where the vortex was unburst.

Thus, the mathematical model of aerodynamic lift coefficient for longitudinal motion of a delta wing can be written as follows:

$$\begin{aligned} \tau_1 \frac{dx}{dt} + x &= x_0(\alpha - \tau_2\dot{\alpha}) \\ C_L &= C_L^{nl}(\alpha, x) + C_{L_q}^{att}\frac{qc}{V} + C_{L_{\dot{\alpha}}}^{att}\frac{\dot{\alpha}c}{V} \end{aligned} \qquad (6)$$

where constant aerodynamic derivatives $C_{L_q}^{att}$ and $C_{L_{\dot{\alpha}}}^{att}$ were introduced to take into account unsteady effects of flow without vortex breakdown ($x \geq 1$). Equation (6), without terms $C_{L_q}^{att}$ and $C_{L_{\dot{\alpha}}}^{att}$, outlines the unsteady effects only in the angle-of-attack region where the $(dx_0/d\alpha) \neq 0$, therefore, we need to introduce attached flow terms to describe the unsteady effects at low angles of attack in a traditional way.

It is well known that in the range of angle of attack with vortex breakdown a significant dependency of unsteady aerodynamic derivatives on amplitude and frequency of oscillations occurs.

The proposed mathematical model can be used to explain this phenomenon. In the case of delta wing pitch oscillations with small amplitude, a linearized form of the mathematical model, Eq. (6), can be derived.

For a small deviation of the dynamic breakdown point position from the steady-state one $\xi = x(t) - x_0[\alpha(t)]$ the above equations can be rewritten as follows:

$$\tau_1 \frac{d\xi}{dt} + \xi = -(\tau_1 + \tau_2)\frac{dx_0}{d\alpha}\dot{\alpha}$$

$$C_L = C_L^{st}(\alpha) + \frac{\partial C_L}{\partial x}\xi + C_{L_q}^{att}\frac{qc}{V} + C_{L_{\dot{\alpha}}}^{att}\frac{\dot{\alpha}c}{V}$$

Resorting to operator form $(d/dt) = s$, it is possible to eliminate the variable ξ, i.e.,

$$C_L = C_L^{st}(\alpha) + K(\alpha)\frac{s\alpha}{\tau_1 s + 1} + C_{L_q}^{att}\frac{qc}{V} + C_{L_{\dot{\alpha}}}^{att}\frac{\dot{\alpha}c}{V} \qquad (7)$$

where $C_L(\alpha)$ represents the steady dependence of the lift coefficient on angle of attack, $C_{L_q}^{att}$ is the damping derivative corresponding to the attached flow at small angles of attack, τ_1 is a relaxation time constant, and s is the Laplace variable. The function $K(\alpha)$ depends on the partial derivative $(\partial C_L/\partial x)$ and on the slope $(dx_0/d\alpha)$

$$K(\alpha) = -(\tau_1 + \tau_2)\frac{\partial C_L}{\partial x}\frac{dx_0}{d\alpha}$$

Thus, for harmonic oscillations of a delta wing

$$\alpha = \alpha_1 + \alpha_s \sin\omega t \qquad q = \dot{\alpha}$$

the aerodynamic lift can be written in the following form:

$$C_L = C_L^{st}(\alpha_1) + C_{L_\alpha}^{fo}\alpha_s \sin\omega t + C_{L_{\dot{\alpha}}}^{fo}\alpha_s\omega \cos\omega t$$

where with the use of Eq. (7)

$$\begin{aligned} C_{L_\alpha}^{fo}(\alpha, \omega) &= C_{L_\alpha}^{st} + K(\alpha)\frac{\omega^2\tau_1}{1 + \omega^2\tau_1^2} \\ C_{L_{\dot{\alpha}}}^{fo}(\alpha, \omega) &= C_{L_q}^{att} + C_{L_{\dot{\alpha}}}^{att} + \frac{K(\alpha)}{1 + \omega^2\tau_1^2} \end{aligned} \qquad (8)$$

Figure 4 shows the dependencies of the lift coefficient out-of-phase derivatives for a delta wing $A = 1.5$ obtained in forced oscillations in a wind tunnel[19] for different frequencies ($\omega = 0.015V/c$—circles, $\omega = 0.144V/c$—squares, $\omega = 0.306 V/c$—triangles) and their approximations (solid lines) by mathematical model, Eq. (7). This level of fitting was achieved for the next value of the relaxation time constant $\tau_1 \approx 15c/V$.

Thus, using one function $K(\alpha)$ and one time constant τ_1 it is possible to describe numerous experimental dependencies obtained at different oscillations frequencies.

It should be noted that the value of the time constant τ_1, obtained from water-tunnel experiments based on the vortex flow visualization results ($\approx 1.5c/V$), differs from its value,

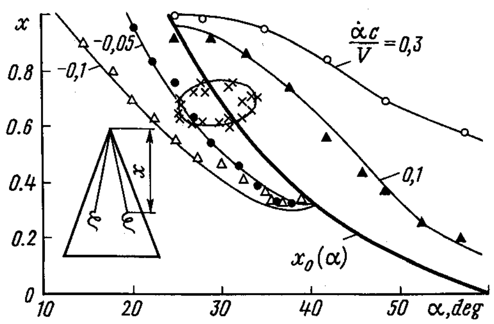

Fig. 3 Experimental data[16] and mathematical modeling results for a delta wing $A = 1.5$ vortex breakdown position.

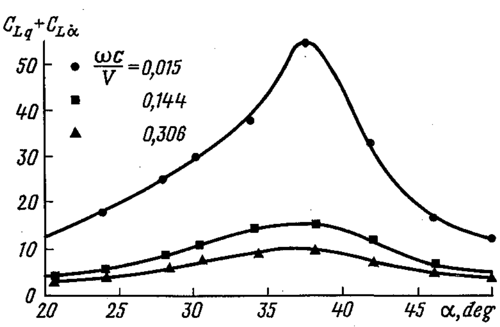

Fig. 4 Experimental data[19] and mathematical fitting results for out-of-phase pitch forced oscillations derivative of a delta wing $A = 1.5$.

obtained from the wind-tunnel forced oscillations results ($\approx 15c/V$). This discrepancy unfortunately cannot be explained on the basis of the available experimental information.

Roll Oscillations

In the case of roll oscillations of the same delta wing (A = 1.5), the breakdown of the right and left vortices will occur at different distances from the apex of the wing. Experimental investigations in a water tunnel[16,20,21] show that in the case of a small sideslip angle it is possible to consider that

$$x_{l,r}(\alpha, \beta) = x_0(\alpha) + \frac{\partial x_{l,r}}{\partial \beta}\beta$$

$$\frac{\partial x_l}{\partial \beta} = -\frac{\partial x_r}{\partial \beta} = K_\beta$$

where K_β is a constant.

In the case of unsteady variations in sideslip, the vortex breakdown point position will, just as during the motion in a longitudinal plane, delay with respect to the position that the vortex breakdown has in steady conditions.[11] By analogy with Eq. (3) one can write the same differential equations describing the variations of the coordinates x_l and x_r:

$$\tau_1 \frac{dx_l}{dt} + x_l = x_0 + K_\beta(\beta - \tau_3\dot{\beta})$$
$$\tau_1 \frac{dx_r}{dt} + x_r = x_0 - K_\beta(\beta - \tau_3\dot{\beta}) \qquad (9)$$

At low values of β, the roll moment induced by the breakdown of vortices above the wing will be approximately proportional to the difference $x = x_l - x_r$. Based on this assumption and Eq. (9) it is possible to write an equation describing a variation of the vortex breakdown induced rolling moment component C_l^{vor}

$$\tau_1 \frac{dC_l^{\text{vor}}}{dt} + C_l^{\text{vor}} = 2\frac{dC_l^{\text{vor}}}{dx}K_\beta(\beta - \tau_3\dot{\beta}) \qquad (10)$$

The total rolling moment during the wing motion with a varying sideslip angle can be presented similarly to the case of the longitudinal motion as

$$C_l = C_l^{\text{vor}} + C_{l_\beta}^{\text{att}}(\alpha)\beta + C_{l_p}^{\text{att}}\frac{pb}{V} + C_{l_{\dot\beta}}^{\text{att}}\frac{\dot{\beta}b}{V} \qquad (11)$$

where the term $C_{l_\beta}^{\text{att}}\beta$ corresponds to the rolling moment component of attached flow and the linear terms $C_{l_p}^{\text{att}}(pb/V)$ and $C_{l_{\dot\beta}}^{\text{att}}(\dot{\beta}b/V)$ define the unsteady aerodynamic load during the rotation without taking into account the vortex breakdown effect. The component C_l^{vor} reflects the influence of vortices breakdown and results in the additional nonlinear unsteady effects.

Equations (10) and (11) are the mathematical model of the form of Eq. (1) for unsteady roll moment of the wing during roll oscillations.

The roll damping coefficient $(C_{l_p} + C_{l_{\dot\beta}} \sin \alpha)^{\text{fo}}$ experimental results for a delta wing with aspect ratio A = 1.5 from Ref. 19 at different angles of attack and frequencies of oscillations ($\omega = 0.030V/b$—circles, $\omega = 0.061V/b$—squares, $\omega = 0.122V/b$—triangles) are presented in Fig. 5.

To simulate these results the linearization of Eqs. (10) and (11) with the kinematic relation $\beta = \varphi \sin \alpha$ taken into account was made similarly to the case of the wing pitch oscillations. The following expression was used to fit the experimental results at different frequencies:

$$(C_{l_p} + C_{l_{\dot\beta}} \sin \alpha)^{\text{fo}} = C_{l_p}^{\text{att}} + C_{l_{\dot\beta}}^{\text{att}} \sin \alpha + \frac{K_p(\alpha)}{1 + \omega^2\tau_1^2} \qquad (12)$$

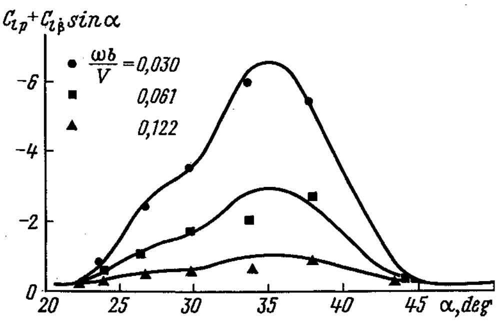

Fig. 5 Experimental data[19] and mathematical fitting results for out-of-phase roll forced oscillations derivative of a delta wing A = 1.5.

where the dependency on oscillation frequency is expressed explicitly in the range of angles of attack where vortex breakdown occurs. The results of fitting are shown in Fig. 5 by solid lines. It is seen that the dependency of the damping coefficient on oscillation frequency for different angles of attack can be described using one function of the angle-of-attack $K_p(\alpha)$, and one time constant $\tau_1 \approx 15c/V$.

The relaxation time constant τ_1, obtained by fitting the above experimental results, is approximately equal to the corresponding value of the time constant $\tau_1 \approx 15c/V$ for pitch oscillation results of the same delta wing. This indicates that in both cases one and the same physical process associated with the development of vortex breakdown above the wing has the considerable influence on unsteady aerodynamic loads of a delta wing.

Mathematical Model for Aircraft Unsteady Aerodynamics at High Angles of Attack

The above simplified examples show that despite considerable differences in flow structures it becomes possible to describe dynamic properties of force and moment variations in a similar way by using the simple first-order differential Eq. (3). This equation describes the time lag features of separated and vortex flow dynamics processes. Yet function g in Eq. (1) for determination of forces and moments in each case was different.

One can also try to use a similar approach for describing the aerodynamic characteristics of an aircraft at high angles of attack. In the complex flow around an aircraft at high angles of attack, both the separation and vortex breakdown phenomena occur. Since the dynamical properties of these phenomena can be described in a similar way, it is reasonable to use Eq. (3) as the simplest approximation also for the full aircraft. The generalized internal state variable in this case would not have the definite physical meaning. The function g in Eq. (1) in this general case must be determined via the adequate identification technique based upon different experimental data.

In the range of small angles of attack the mathematical model has to coincide with the traditional description using the aerodynamic derivatives concept. The influence of unsteady vortex wake, tail wash time lag, etc., will be considered in a linear approximation and independently of the main component generated by separation and vortex flow dynamic effects. The mathematical model for the longitudinal aerodynamic force and moment can be written in the following form:

$$C_Z = C_Z^{\text{nl}}(\alpha, x) + C_{Z_q}^{\text{att}}\frac{q\bar{c}}{V} + C_{Z_{\dot\alpha}}^{\text{att}}\frac{\dot{\alpha}\bar{c}}{V} + C_{Z_{\delta_e}}\delta_e$$

$$C_m = C_m^{\text{nl}}(\alpha, x) + C_{m_q}^{\text{att}}\frac{q\bar{c}}{V} + C_{m_{\dot\alpha}}^{\text{att}}\frac{\dot{\alpha}\bar{c}}{V} + C_{m_{\delta_e}}\delta_e \qquad (13)$$

$$\tau_1 \frac{dx}{dt} + x = x_0(\alpha - \tau_2\dot{\alpha}), \qquad (|x| \leq 1)$$

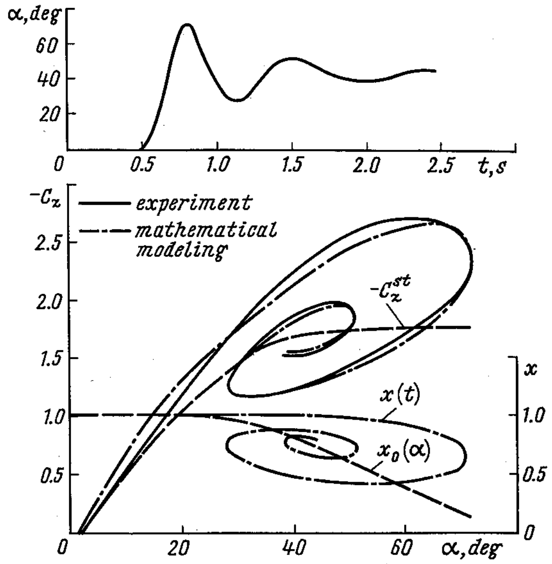

Fig. 6 Results of high amplitude free pitch oscillations of aircraft model in a wind tunnel.

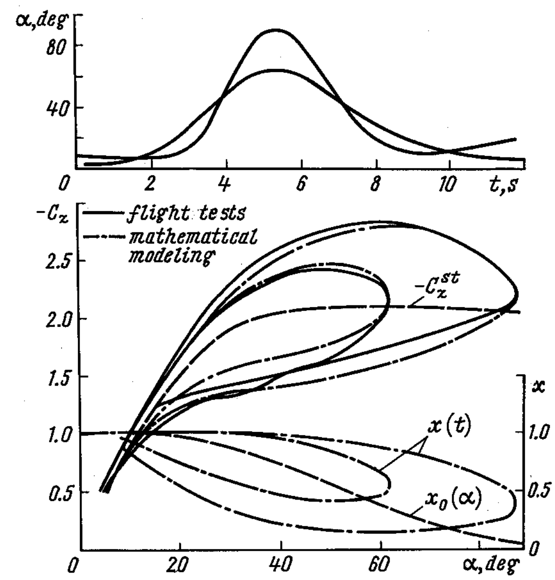

Fig. 7 Flight test results for rapid high angle-of-attack maneuvers.

The nonlinear functions C_Z^{nl}, C_m^{nl}, which give the dependencies of aerodynamic loads on kinematic parameters and an internal variable, are given in the general form. The flow structure about an aircraft at high angles of attack is considerably more complicated, and it is impossible to hope that any analytical dependencies for C_Z^{nl}, C_m^{nl} can be obtained. So they have to be determined by the special identification technique, but this is beyond the scope of this article. The main objective of this work is to demonstrate the capability of the proposed mathematical model.

Equation (13) has to be valid both for disturbed motions with small amplitude oscillations, Eq. (7), and for arbitrary motions including a large amplitude dynamic high angle-of-attack attainment.

In order to simplify the problem of identification of such a mathematical model one can assume that the nonlinear functions C_Z^{nl}, C_m^{nl} can be presented in the following form:

$$\begin{aligned} C_Z^{nl}(\alpha, x) &= C_{Z_1}(\alpha)g(x) + C_{Z_2}(\alpha)[1 - g(x)] \\ C_m^{nl}(\alpha, x) &= C_{m_1}(\alpha)g(x) + C_{m_2}(\alpha)[1 - g(x)] \end{aligned} \tag{14}$$

where $C_{Z_1}(\alpha)$, $C_{m_1}(\alpha)$ and $C_{Z_2}(\alpha)$, $C_{m_2}(\alpha)$ are, respectively, the dependencies of the coefficients C_Z and C_m in the two limit cases $x = 1$ and $x = 0$. The functions $C_{Z_1}(\alpha)$, $C_{m_1}(\alpha)$ are the envelope curves for a set of C_Z and C_m variations with different positive pitch rates. The similar functions $C_{Z_2}(\alpha)$, $C_{m_2}(\alpha)$ are the envelope curves for cases with different negative pitch rates. The function $g(x)$ plays the role of a normalized weight function $g(x) \in [0, 1]$, $x \in [0, 1]$ which permits the expression of the simplified dependency on x. The following expression for this function was used $g(x) = x + ax(x - 1)$, where the constant $a \in [-1, 1]$ was determined by the least squares technique.

The dynamic effects of separated and vortex flow are especially significant during high angle-of-attack attainment with large amplitudes. To investigate the unsteady aerodynamic characteristics of a fighter aircraft, special wind-tunnel experiments were conducted.[8] The aircraft model was mounted on a cardan joint with 1 DOF in pitch. During transient motion from small values of angle of attack to the trim position at high angle of attack, the total aerodynamic force acting on the aircraft model was measured. The typical angle-of-attack time history is shown in the upper part of Fig. 6.

The variations of the total vertical force coefficient in steady and unsteady conditions are shown on the lower part of Fig. 6 by solid lines. At small angles of attack, the variation of the vertical force coefficient differs slightly from that observed in steady-state conditions. When high angles of attack are attained the dynamic value of the coefficient C_Z exceeds considerably its steady-state values. When α_{max} is attained ($\dot{\alpha} = 0$) the value of C_Z also exceeds the value of $C_Z^{st}(\alpha_{max})$, which indicates that there is a considerable time lag effect due to the flow structure adjustment. The last feature, e.g., cannot be described using the aerodynamic derivatives.

The results of the mathematical modeling of the vertical force coefficient and x variations using Eqs. (13) and (14) are presented in Fig. 6 by dashed lines for static conditions and dot-dashed lines for unsteady conditions. The above example shows that the proposed mathematical model takes into account the time lag effects of separated and vortex flow adjustment.

The next example shows the flight tests data,[8] obtained during the well-known fighter aircraft maneuver, when the angle of attack achieves very large values up to 90 deg for a very short time. The angle-of-attack time histories with different amplitudes are presented in the upper part of Fig. 7. The appropriate transient variation of the vertical force coefficient vs angle of attack (solid lines) was obtained after processing the measured in flight kinematic parameters such as velocity, angle of attack, and normal acceleration.[22] The dot-dashed lines show the results of mathematical modeling of the vertical force coefficient and internal variable dependencies in transient motions. The dashed lines show the obtained corresponding static dependencies.

Thus, the proposed mathematical model, as the above results show, is suitable for the description of the different dynamic maneuvers with attainment of high angles of attack $\alpha_{max} \simeq 60 \div 90$ deg.

Concluding Remarks

The proposed mathematical model including the state-space representation of separated and vortex flow can describe the different unsteady effects observed in experimental investigations. In particular, it is possible to explain the dependence of aerodynamic characteristics on motion prehistory, and the influence of reduced frequency and oscillation amplitude on unsteady aerodynamic derivatives, obtained in forced oscillation experiments.

Use of the mathematical model for the unsteady aerodynamics, based on the first-order ordinary differential equations, gives an easy way for flight dynamics simulation by appending these equations to the equations governing the aircraft motion.

The stability analysis and dynamic simulation of aircraft motion at high angles of attack can then be done either by eigenvalue analysis or by the time integration of the governing equations using standard numerical algorithms.

The presented results are the initial steps in the development of this mathematical modeling method. These results demonstrate the method's ability to model different complex unsteady aerodynamic phenomena that take place at high angles of attack.

The form of the mathematical model can be more complicated. If necessary, the right side of Eq. (3) can be written in a more general form $x_0(\alpha, \dot{\alpha}, q, \ldots)$. When the aerodynamic hysteresis occurs, Eq. (3) can be rewritten in the nonlinear form: $\dot{x} = F(x, \alpha, \dot{\alpha}, q, \ldots)$, where the function F in steady conditions $F(x, \alpha, 0, q, \ldots) = 0$ gives, e.g., the bistable solutions for the internal variable. The order of the differential operator can be increased to describe more complex flow adjustment processes.

Utilization of the proposed mathematical model is closely related to the problem of the identification of its structure and evaluation of unknown parameters. The special efforts are needed to develop the adequate methods.

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